NOTTINGHAM LIE ALGEBRAS WITH DIAMONDS OF FINITE AND INFINITE TYPE

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ABSTRACT. We consider a class of infinite-dimensional, modular, graded Lie algebras whose simplest example is the Lie algebra associated to the Nottingham group with respect to its lower central series. We identify two subclasses of Nottingham algebras as loop algebras of simple Lie algebras of Hamiltonian type. A property of Laguerre polynomials of derivations plays a crucial role in the given constructions.

1. Introduction

An infinite-dimensional graded Lie algebra $L = \bigoplus_{k=1}^{\infty} L_k$, over a field $\mathbb{F}$ of characteristic $p > 0$, is said to be thin if the first homogeneous component L_1 has dimension two and the following *covering property* holds

$$L_{i+1} = [u, L_1] \quad \text{for all } 0 \neq u \in L_i, \text{ for all } i \geq 1.$$

Thin algebras are the Lie theoretic counterpart of thin pro p-groups, or groups of *width two* and *obliquity zero* using the language of [KLGP97], and they were originally introduced in [CMNS96] as Lie algebras associated to thin pro p-groups with respect to their lower central series. The definition of thin algebra implies that L is generated by L_1 as Lie algebra and that every homogeneous component has dimension one or two. We call *diamond* any homogenous component of dimension two, so that L_1 is the first diamond of L. If this is the only one in L then the algebra has *maximal class* and these algebras have been completely classified in [CN00] and [Jur05]. A systematic investigation of thin Lie algebras with the second diamond in degree $q = p^n$ was initiated in [CM04], where they were called *Nottingham (Lie) algebras*, because the simplest example is the graded Lie algebra associated with the lower central series of the Nottingham group (see [Jen54, Joh88, Cam00]). One can assign a *type* to each diamond later than the first, which takes value in the underlying field plus infinity, but the second diamond of a Nottingham Lie algebra has always type -1. Diamonds of type zero or one are really one-dimensional components, but it is convenient to allow them in certain positions and dub them *fake*. A first convenience in allowing fake diamonds is that the

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definition of Nottingham algebra extends to the case of even characteristic, where the second diamond is a fake one of type $1 \equiv -1 \bmod 2$.
Various diamond patterns are possible and in Section 2 we dispose all the possibilities. In the present paper we construct Nottingham algebras having diamonds of both finite and infinite types. Our thin algebras have diamonds in all degrees congruent to 1 modulo $q-1$. The diamonds occur in sequences of p^s-1 diamonds of infinite type separated by single occurrences of diamonds of finite type, and the latter types follow an arithmetic progression. The case the arithmetic progression is the constant sequence -1 has been considered in [AM07]. The remaining algebras fall in two subclasses according with the fact the type μ of the second diamond of finite type in order of occurrence, that is the diamond in degree $(p^s+1)(q-1)+1$, belongs or not to the prime field. In Section 5 we explicitely construct the algebras under consideration as *loop* algebras of an Hamiltonian simple Lie algebra H of dimension p^n-2 in the first case and p^n in the second one. The crucial step in the construction procedure is to produce a suitable grading of H over the cyclic group of order $(q-1)p^{s+1}$. To realize the latter, we apply a general result on (generalized) Laguerre polynomials of derivations, which is described in [AM12] and that we briefly recall in Section 3. This general principle specializes, for nilpotent derivations, to a property of the Artin-Hasse exponential of a derivation, exposed in [Mat05], and that we still recall in Section 3 in the particular instance when the Artin-Hasse exponential coincides with the truncated exponential $\sum_{i=0}^{p-1} \frac{x^i}{i!}$. In the present paper we apply the mentioned result on Laguerre polynomials in the case μ does not belong to the prime field and its specialization in terms of exponential series in the case μ belongs to the prime field. The results of this paper are valid in any positive characteristic. However we mention that C. Scarbolo in his master's thesis [Sca10] has given an explicit construction of one subclass of Nottingham Lie algebras with diamonds of both finite and infinite type in characteristic two, still as loop algebras of an Hamiltonian algebra H. His method relies on direct inspection of H, taking advantage on some peculiarities of the characteristic two case.
Extensive machine computations suggest that the Nottingham algebras with diamonds of both finite and infinite type are only those we have described above.

2. Nottingham algebras

Let $L = \bigoplus_{i=1}^{\infty} L_i$ be a Nottingham algebra over the field $\mathbb{F}$ of characteristic $p > 0$, with second diamond in degree $q = p^n$, for some $n > 0$. We assume $q > 3$ postponing the cases $p = 3 = q$ and $p = 2$ at the end of this section. It follows that $\dim(L_3) = 1$ and hence there exists a nonzero $y \in L_1$ such that $[L_2, y] = 0$. According to [CJ99] the quotient L/L^q is metabelian, that is y centralizes the homogeneous components $L_2, \dots L_{q-2}$. It follows from [Car97] (but see also [AJM10] for a more extensive discussion on the structure of Nottingham algebras

up to their second diamond) that one can choose $x \in L_1 \setminus \mathbb{F}y$ such that $[vxx] = 0 = [vyy]$ and $[vyx] = -2[vxy]$ for any non-trivial element v in degree $q-1$. We remark that the generators x and y of L are determined up to a scalar multiple. The relations between the generators $[vxx]$, $[vxy]$, $[vyx]$ and $[vyy]$ we have in degree $q+1$ are a special case of a more general phenomenon that occurs for any diamond of L past the first. Suppose in fact that L_h is a diamond of L, with $h > 1$. It is proved in [Mat] that a thin Lie algebra cannot have two consecutive diamonds, provided the third homogeneous component is one-dimensional. Therefore $\dim(L_{h-1}) = 1 = \dim(L_{h+1})$ and let w be a generator for L_{h-1}, thus the diamond in degree h is spanned by $[wx]$ and $[wy]$. If

$$[wxx] = 0, \quad [wyy] = 0, \quad \mu[wyx] = (1-\mu)[wxy] \tag{2.1}$$

for some $\mu \in \mathbb{F}$ then the diamond L_h is said to be *of (finite) type* μ. In particular the second diamond L_q has type -1. This terminology extends to the case of diamond *of infinite type* $\mu = \infty$ by reading the third equation in 2.1 as $[wyx] = -[wxy]$. We stress that the type of a diamond is not affected by replacing the generators x and y by their scalar multiples.

As mentioned in the Introduction, strictly speaking diamonds of type zero or one cannot occur. In fact a diamond L_h of type zero should satisfy $[wxx] = 0 = [wxy]$ and therefore the element $[wx]$ would be central, that is trivial because of the covering property and the assumption that L is infinite dimensional. Similarly if L_h is a diamond of type one, then the element $[wy]$ must be trivial, since $[wyx] = 0 = [wyy]$ in this case. This contradicts the assumption that L_h has dimension two. Nevertheless we allow certain one-dimensional homogenous component of L to be *fake* diamonds of type 0 or 1, by relaxing the assumption that a diamond has dimension 2. Therefore a diamond of finite type μ is a homogeneous component L_h such that L_{h-1} is spanned by a single element w which satisfies the corresponding relations 2.1. We will speak of *genuine* diamond when $\dim(L_h) = 2$.

We now describe the possibilities for a third genuine diamond of a Nottingham algebra L. From a result in [CM04], refined in [You01], follows that a third genuine diamond of L cannot occur before than degree $2q-1$, provided $q \neq 5$. This result is the best possible since a counterexample in the case $p = q = 5$, in which a third diamond occurs in degree $2q-1 = 7$, can be found in [CMNS96]. Although the choice of terminology is really a matter of convenience, this algebra does not share several common features of Nottingham algebras. Therefore we exclude it by requiring that a Nottingham algebra satisfy the condition $\dim(L_7) = 1$ for $p = q = 5$.

Nottingham algebras with third genuine diamond (if any) in degree greater than $2q-1$ have been completely classified in [You01]. Some of them can be recognized as Nottingham algebras with a third fake diamond in degree $2q-1$ and they were originally described in [CM04]. The remaining ones belong to three countable families, two of which consist of soluble algebras having only two diamonds, or to

an uncountable family which is in one-to-one correspondence with a subclass of the graded Lie algebras of maximal class.

The case L has a genuine diamond of finite type μ in degree $2q-1$ was investigated in [CM04] for $p > 5$. The algebra L turns out to be uniquely determined by a certain finite-dimensional graded quotient of it, it has diamond in each degree congruent to 1 modulo $q-1$, the types of the diamonds follow an arithmetic progression (counting also the fake ones), which is thus determined by μ. In the case the third diamond is a fake one, that is $\mu \in \{0, 1\}$, a similar result holds, once some extra conditions are prescribed.

We recall at this point the definition of *loop* algebra which is crucial in the exposition that follows and for the purposes of this paper.

Definition 2.1. Let S be a finite-dimensional Lie algebra over the field $\mathbb{F}$ and with a cyclic grading $S = \bigoplus_{k\in\mathbb{Z}/N\mathbb{Z}} S_k$, let U be a subspace of $S_{\bar{1}}$ and let t be an indeterminate over $\mathbb{F}$. The *loop* algebra of S (with respect to the given cyclic grading and the subspace U, that we tacitly omit whenever U coincides with the whole $S_{\bar{1}}$) is the Lie subalgebra of $S \otimes \mathbb{F}[t]$ generated by $U \otimes t$.

Some of the algebras described in [CM04], namely, those where the arithmetic progression is the constant sequence -1, were produced in [Car97] as (twisted) loop algebras of Zassenhaus algebras $W(1; n)$. The remaining Nottingham algebras with all diamonds of finite type fall in two subclasses. Those where the type of the third diamond (and hence all diamond types) belongs to the prime field but is different from -1 and those were the type of the third diamond do not belong to the prime field. The former were constructed in [Avi02] as loop algebras of graded Hamiltonian algebras $H(2; (1, n))^{(2)}$ (suitably extended by an outer derivation in a couple of cases), the latter in [AM07] as loop algebras of Hamiltonian algebras $H(2 : (1, n); \Phi(1))$. Note that the presence of fake diamonds of both types zero and one in the first case, and the absence of fake diamonds in the second, reflects in the dimension of the simple algebra used in the construction, namely two less than a power of p and a power of p, respectively (for $p > 2$).

The last possibility for the diamond L_{2q-1} is to be of infinite type. A family of Nottingham algebras with third diamond in degree $2q-1$ and all diamonds, but the second one, of infinite type is exhibited in [You01]. Another possibility is considered in [AM07] Section 5, where Nottingham algebras with diamonds occurring at regular intervals, the types pattern being isolated occurrences of diamonds of type -1 separated by p^s-1 diamonds of type ∞, for some $s > 0$, were produced, as loop algebras of Hamiltonian algebras $H(2 : (s+1, n); \Phi(1))$. The diamond structure of the latter algebras turns out to be a special case of a more general diamond pattern we consider in this paper. Precisely for any fixed $s > 0$ we consider Nottingham algebras L with diamonds in all degrees $t(q-1)+1$, $t \geq 1$. If $t \not\equiv 1 \bmod p^s$, the corresponding diamond has infinite type while for $t \equiv 1 \bmod p^s$ the corresponding diamond has finite type. The type of the finite

diamonds follow an arithmetic progression. Let μ be the type of the diamond in degree $(p^s+1)(q-1)+1$, that is μ is the type of the second finite diamond in order of occurrence. Then for $t \equiv 1 \bmod p^s$, say $t = 1 + rp^s$, the diamond in degree $t(q-1)+1$ has type $r\mu + r - 1$. When μ belongs to the prime field, the type of each finite diamond is the prime field, while for $\mu \in \mathbb{F} \setminus \mathbb{F}_p$ the sequence of the types of the finite diamonds is not contained in $\mathbb{F}_p$. The case $\mu = -1$, that is when each diamond of finite type has type -1, is therefore the case considered in [AM07] Section 5. The purpose of this paper is to explicitly costruct Nottingham algebras with diamonds of both finite and infinite type as described above. The costruction is given in Section 5. Note also that allowing $s = 0$ in the above description one obtains a Nottingham algebra with all diamonds of finite type as described in [CM04]. However, the constructions given in [Avi02] and [AM07] are not a special case of the constructions given in the present paper.

Remark 2.2. The case $p = 3 = q$ comprises a host of thin algebras and we refer the reader to [AM07] for a detailed description of all the known possibilities. However in order that the results of this paper make sense also in this case, we extend the definition of diamond type given by relations 2.1 to the present situation (even if the characterization $[L_2, y] = 0$ of the distinguished generator y fails here, because $C_{L_1}(L_2) = \{0\}$).

Remark 2.3. In even characteristic the second diamond of a Nottingham algebra L becomes fake of type $1 \equiv -1 \mod 2$ causing an ambiguity in the definition of the second diamond of a thin algebra as the next homogeneous component of dimension two after L_1. Moreover a Nottingham algebra in characteristic two can have all diamonds of types in the prime field $\mathbb{F}_2 = \{0, 1\}$ hence all diamonds are fake ones and thus it is actually a Lie algebra of maximal class. Apart for these peculiarities the definition of type of a diamonds extends to the case of characteristic 2 and all the contructions in [Car97], [Avi02] and [AM07] mentioned above make sense for $p = 2$.

3. Laguerre polynomials of derivations and gradings of algebras

In this section we recall two methods to construct new gradings of a non-associative algebra over a field of prime characteristic, from a given one. The first one relies on a property of the Artin-Hasse exponential of a nilpotent derivation and it is described in [Mat05]. In the present paper we deal with a special instance of the more general principle on the Artin-Hasse series, namely when a derivation D satisfies the condition D^p and the Artin-Hasse series coincides with the exponential series $\exp(D) = \sum_{i=0}^{p-1} \frac{D^i}{i!}$. We quote from [Mat05] the following theorem

Theorem 3.1. *Let A be a non-associative algebra over a field of prime characteristic p, graded over the integers modulo m. Suppose that A has a nilpotent*

derivation D graded of degree d, with $m \mid pd$. Then the direct sum decomposition $A = \oplus_i \exp(D)A_i$ is a grading over the integers modulo m.

The second method relies on a property of (generalized) Laguerre polynomials of a derivation and it is stated in [AM12]. The classical (generalized) Laguerre polynomial of degree $n \geq 0$ is defined as

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{\alpha + n}{n-k} \frac{(-x)^k}{k!},$$

where α is a complex number. Instead of $L_n^{(0)}(x)$ it is usual to write $L_n(x)$. If $\mathbb{F}$ is a field of characteristic $p > 0$, the polynomial $L_{p-1}^{(\alpha)}(x)$ for $\alpha \in \mathbb{F}$ can be regarded as a polynomial in $\mathbb{F}[x]$. Explicitly we have

$$L_{p-1}^{(\alpha)}(x) = \sum_{k=0}^{p-1} \binom{\alpha + p - 1}{p-1-k} \frac{(-x)^k}{k!}$$

and, in the special case $\alpha = 0$, the Laguerre polynomial $L_{p-1}(x)$ coincides with the *truncated* esponential $\mathrm{E}(x)$ of x, where $\mathrm{E}(x) = \sum_{k=0}^{p-1} \frac{x^k}{k!}$. Laguerre polynomials $L_{p-1}^{(\alpha)}(D)$ of a derivation D of a non-associative algebra A over $\mathbb{F}$ can be used to produce a grading of a A from a given one. We quote from [AM12] the following theorem

Theorem 3.2. *Let $A = \bigoplus_k A_k$ be a non-associative algebra over the field $\mathbb{F}$ of characteristic $p > 0$, graded over the integers modulo m. Suppose that A has a graded derivation D of degree d such that $D^{p^2} = D^p$, with $m \mid pd$. Suppose that $\mathbb{F}$ contains the field of p^p elements, and choose $\gamma \in \mathbb{F}$ with $\gamma^p - \gamma = 1$. Let $A = \bigoplus_{a \in \mathbb{F}_p} A^{(a)}$ be the decomposition of A into a direct sum of eigenspaces for D^p, and let $L_D : A \to A$ be the linear map whose restriction to $A^{(a)}$ coincides with $L_{p-1}^{(a\gamma)}(D)$. Then $A = \bigoplus_k L_D(A_k)$ is also a grading of A over the integers modulo m.*

Let now D be a derivation of A such that $D^{p^2} = D^p$ and let $A = \bigoplus_{a \in \mathbb{F}_p} A^{(a)}$ be the decomposition of A into a direct sum of eigenspaces for D^p. For $\lambda \in \mathbb{F}$ consider the derivation $\lambda D \in \mathrm{Der}(A)$. Suppose that there exists $\pi \in \mathbb{F}$ with $\pi^p - \pi = \lambda^p$. The theorem above implies that, set $L_{\lambda D} : A \to A$ the linear map whose restriction to $A^{(a)}$coincides with $L_{p-1}^{(a\pi)}(\lambda D)$, then $A = \bigoplus_k L_{\lambda D}(A_k)$ is also a grading of A over the integers modulo m.

4. Preliminaries on some Lie algebras of Cartan type

In this section we briefly recall the definition and some gradings of the Zassenhaus algebra and certain Lie algebras of Cartan type belonging to the Hamiltonian series we will use in the paper, referring the reader to [Str04] and [CM05] for a

general reference and to [AM07] for a broader discussion on some adjustements we make with respect to [Str04].

Let $\mathbb{F}$ be an arbitrary field of prime characteristic p, and n a positive integer. The algebra $\mathcal{O}(1,n)$ of *divided powers in one indeterminate* x *of height* n is the $\mathbb{F}$-vector space of formal linear combinations $\sum_i a_i\, x^{(i)}$ where $a_i \in \mathbb{F}$, $0 \leq i \leq p^n - 1$, with multiplication defined by $x^{(i)} \cdot x^{(j)} = \binom{i+j}{i} x^{(i+j)}$ on monomials, and extended by linearity and by postulating commutativity and associativity of the multiplication. The algebra $\mathcal{O}(2,(n_1,n_2))$ of *divided powers in two indeterminates* x *and* y *of heights* n_1 *and* n_2 *respectively* is isomorphic with the tensor product $\mathcal{O}(1;n_1) \otimes \mathcal{O}(1;n_2)$ and can be described as the $\mathbb{F}$-vector space of formal linear combinations $\sum_i a_{i,j}\, x^{(i)} y^{(j)}$ where $a_{i,j} \in \mathbb{F}$, $0 \leq i \leq p^{n_1} - 1$ and $0 \leq j \leq p^{n_2} - 1$, with multiplication defined by $x^{(i)} y^{(j)} x^{(k)} y^{(l)} = \binom{i+k}{i}\binom{j+l}{j} x^{(i+k)} y^{(j+l)}$ on monomials, and extended by linearity and by postulating commutativity and associativity of the multiplication. We use the standard shorthands $\bar{x} = x^{(p^{n_1}-1)}$ and $\bar{y} = y^{(p^{n_2}-1)}$. In the algebra $\mathcal{O}(1,1)$ we define, for any element $\alpha \in \mathbb{F}$, $(1+x)^\alpha = \displaystyle\sum_{i=0}^{p-1} \binom{\alpha}{i} i! x^{(i)}$. Note that whenever $\alpha = a \in \mathbb{F}_p$ the expression above specializes to the usual binomial theorem expressed in the divided powers. We also have the natural rule of multiplication $(1+x)^\alpha (1+x)^\beta = (1+x)^{\alpha+\beta}$ since $(1+x)^\alpha(1+x)^\beta = \sum_{k=0}^{p-1} \left(\sum_{i=0}^{k} \binom{\alpha}{i}\binom{\beta}{k-i} \right) k! x^{(k)}$.

The *Zassenhaus algebras* $W(1;n)$ is the Lie algebra of *special derivations* of $\mathcal{O}(1,n)$, that is those of the form $D = f\,\partial$ with $f \in \mathcal{O}(1,n)$ where $\partial(x^{(i)}) = x^{(i-1)}$ for $i > 0$ and (necessarily) $\partial(x^{(0)}) = 0$. In odd characteristic it is a simple Lie algebra of dimension p^n and it has a standard $\mathbb{Z}$-grading $W(1,n) = \bigoplus_{i=-1}^{p^n-2} W(1,n)_i$ where the homogeneous component $W(1;n)_i$ is one-dimensional, generated by $E_i = x^{(i+1)}\partial$, for $i = -1, \dots, p^n - 2$. Direct computation shows that

$$[E_i, E_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i} \right) E_{i+j}. \tag{4.1}$$

In particular, we have $[E_{-1}, E_j] = E_{j-1}$, and $[E_0, E_j] = jE_j$. The Zassenhaus algebra has also a grading over (the additive group of) $\mathbb{F}_{p^n}$, that we do not need in this paper. In characteristic two the Zassenhaus algebra $W(1;n)$ is not simple, but its derived subalgebra $W(1;n)^{(1)} = \langle E_i \mid i \neq p^n - 2 \rangle = \langle e_\alpha \mid \alpha \neq 0 \rangle$ is simple. The Hamiltonian algebra $H(2;(n_1,n_2))^{(2)}$, which is a simple Lie algebra of dimension $p^{n_1+n_2} - 2$ for $p > 2$, can be identified with the subspace

$$\mathrm{span}\{x^{(i)} y^{(j)} \mid 0 < (i,j) < (p^{n_1}, p^{n_2})\}$$

of $\mathcal{O}(2;(n_1,n_2))$ with the *Poisson bracket*

$$\begin{aligned}\{x^{(i)}y^{(j)},x^{(k)}y^{(l)}\} &= x^{(i)}y^{(j-1)}x^{(k-1)}y^{(l)} - x^{(i-1)}y^{(j)}x^{(k)}y^{(l-1)} \\ &= N(i,j,k,l)\,x^{(i+k-1)}y^{(j+l-1)},\end{aligned} \tag{4.2}$$

where

$$N(i,j,k,l) := \binom{i+k-1}{i}\binom{j+l-1}{j-1} - \binom{i+k-1}{i-1}\binom{j+l-1}{j}.$$

In characteristic two, $H(2;(n_1,n_2))^{(2)}$ is simple provided $n_1 > 1$ and $n_2 > 1$. The algebra $H(2:(n_1,n_2);\Phi(1))$ is isomorphic to $\mathcal{O}(2;(n_1,n_2))$ with respect to the Lie bracket

$$\{x^{(i)}y^{(j)},x^{(k)}y^{(l)}\} = N(i,j,k,l)\,x^{(i+k-1)}y^{(j+l-1)} \quad \text{if } i+k>0, \text{ and} \tag{4.3}$$

$$\{y^{(j)},y^{(l)}\} = \left(\binom{j+l-1}{l} - \binom{j+l-1}{j}\right)\bar{x}y^{(j+l-1)}. \tag{4.4}$$

It follows from Equations (4.3) and (4.4) that the algebra $H(2;(n_1,n_2);\Phi(1))$ is graded over the group $\mathbb{Z}/p^{n_1}\mathbb{Z}\times\mathbb{Z}$ by assigning degree $(i+p^{n_1}\mathbb{Z},j)$ to the monomial $x^{(i+1)}y^{(j+1)}$. In characteristic two $H(2;(n_1,n_2);\Phi(1))$ is not simple. In fact, its derived subalgebra is given by the same description given above for $H(2;(n_1,n_2);\Phi(1))$ but with $0\le(i,j)<(p^{n_1},p^{n_2})$. It is easy to prove that $H(2;(n_1,n_2);\Phi(1))^{(1)}$ is simple, of dimension $2^{n_1+n_2}-1$.
We recall from [AM07] Section 5.1 that the algebra $H=H(2;(n_1,n_2);\Phi(1))$ in odd characteristic has a grading $H=\bigoplus_{k\in\mathbb{Z}/N\mathbb{Z}}H_k$, where $N=p^{n_1}(q-1)$ and $q=p^{n_2}$, obtained by assigning degree $-(q-1)i-j+N\mathbb{Z}$ to the monomial $x^{(i+1)}y^{(j+1)}$. Each homogeneous component is one-dimensional, except those of degree k with $k\equiv 1 \mod (q-1)$ which are two-dimensional. In particular H_1 is generated by x and $\bar{y}$. The loop algebra of H with respect to this grading (and taking as subspace U the whole homogenous component H_1) is a Nottingham algebra with diamonds in all degrees congruent to 1 modulo $q-1$. The diamonds have all type ∞ except those in degree congruent to q modulo $p^{n_1}(q-1)$ which have type -1. A similar result holds for $p=2$ apart that x and $\bar{y}$ generate the derived subalgebra $H(2;(n_1,n_2);\Phi(1))^{(1)}$.

5. Nottingham algebras with diamonds of finite and infinite type

In this section we will construct Nottingham Lie algebras with p^s-1 diamonds of infinite type separated by single occurrences of a diamond of finite type. We consider first the case the types of the finite diamonds do not belong to the field with p elements, a part for the diamonds of type -1. The ground field $\mathbb{F}$ is prescribed to contain at least the various finite diamond types. However we will not make assumption on $\mathbb{F}$ in advance, allowing ourselves to enlarge the field when necessary.

We assume for now that p is odd, deserving the Remark 5.2 to the case $p = 2$. Let $H = H(2; (s + 1, n); \Phi(1))$ for some fixed s and n positive integers and set $q = p^n$ and $N = p^{s+1}(q - 1)$. Let D be the derivation $(\operatorname{ad} y)^{p^s}$ which acts on the monomial $x^{(ap^s)}x^{(k+1)}y^{(j+1)}$, for $a \in \mathbb{F}_p$, $-1 \leq k \leq p^s - 2$ and $-1 \leq j \leq q - 2$, following the rule

$$D(x^{(ap^s)}x^{(k+1)}y^{(j+1)}) = \begin{cases} x^{((a-1)p^s)}x^{(k+1)}y^{(j+1)} & \text{for } a > 0 \\ -jx^{((p-1)p^s)}x^{(k+1)}y^{(j+1)} & \text{for } a = 0 \end{cases}$$

It follows that D^p acts semisimply on H with eigenvalues the elements in the prime field

$$D^p(x^{(ap^s)}x^{(k+1)}y^{(j+1)}) = -jx^{(ap^s)}x^{(k+1)}y^{(j+1)}.$$

Consider the grading of $H = \bigoplus_l H_l$ over the cyclic group of order N, mentioned at the end of the previous section, obtained by assigning degree $-(q-1)(ap^s+k)-j+N\mathbb{Z}$ to the monomial $x^{(ap^s)}x^{(k+1)}y^{(j+1)}$. The derivation D turns out to be graded of degree N/p with respect to this grading. Note also that each homogeneous element in the grading is an eigenvector for D^p. Let $\sigma \in \mathbb{F} \setminus \{0\}$ and suppose there exists $\pi \in \mathbb{F}$ such that $\pi^p - \pi = \frac{1}{\sigma^p}$. We can therefore apply the result of Section 3 with derivation $\frac{D}{\sigma}$, obtaining a grading of H still over the integers modulo N. Explicitly we are considering the elements

$$\bar{e}_{j,k,a} = L_{p-1}^{(-j\pi)}\left(\tfrac{D}{\sigma}\right)(x^{(ap^s)}x^{(k+1)}y^{(j+1)})$$

for $j = -1, \cdots q - 2$, $k = -1, \cdots p^s - 2$ and $a \in \mathbb{F}_p$. The argument above ensures that they constitute a basis of H, graded over the integers modulo N by assigning to $\bar{e}_{j,k,a}$ degree $-(q-1)(ap^s + k) - j + N\mathbb{Z}$. Each homogeneous component in this grading has dimension one, except those in degree congruent to 1 modulo $q - 1$ which are two-dimensional. For later convenience we multiply $\bar{e}_{j,k,a}$ by the the coefficient $c_{k,k,a} = a!\sigma^a\binom{-j\pi+a}{a}\binom{-j\pi+p-1}{p-1}^{-1}$, obtaining the elements

$$e_{j,k,a} = c_{j,k,a}\bar{e}_{j,k,a} = (1 + \sigma x^{(p^s)})^{-j\pi+a}x^{(k+1)}y^{(j+1)} \tag{5.1}$$

for $j = -1, \cdots q - 2$, $k = -1, \cdots p^s - 2$ and $a \in \mathbb{F}_p$. It is easy to find the multiplication table of the basis $e_{j,k,a}$. We have

$$\{e_{j,k,a}, e_{l,h,b}\} = \left[\binom{k+h+1}{h}\binom{j+l+1}{j} - \binom{k+h+1}{k}\binom{j+l+1}{l}\right] e_{j+l,k+h,a+b} \tag{5.2}$$

for $k + h > -1$ and

$$\{e_{j,-1,a}, e_{l,-1,b}\} = \sigma\left[\binom{j+l+1}{j}(-l\pi + b) - \binom{j+l+1}{l}(-j\pi + a)\right] e_{j+l,p^s-2,a+b-1}. \tag{5.3}$$

In fact, since the basis is graded we know that the commutator $\{e_{j,k,a}, e_{l,h,b}\}$ is a scalar multiple of $e_{j+l,k+h,a+b}$ for $k + h > -2$. Thus we have only to compute the scalar, for example by computing the coefficient of $x^{k+h+1}y^{j+l+1}$ in the result, noting that $x^{(k+1)}y^{(j+1)}$ always appears with coefficient 1 in $e_{j,k,a}$. To do this is sufficient to compute the product of the only relevant terms, namely

$\{x^{(k+1)}y^{(j+1)}, x^{(h+1)}y^{(l+1)}\}$. Similarly in the case $k+h=-2$, the commutator $\{e_{j,-1,a}, e_{l,-1,b}\}$ is a scalar multiple of $e_{j+l,p^s-2,a+b-1}$. The scalar can be recovered by computing the coefficient of $x^{p^s-1}y^{j+l+1}$ in the result. In this case the product of the relevant terms is

$$\sigma(-j\pi+a)\{x^{(p^s)}y^{(j+1)}, y^{(l+1)}\} + \sigma(-l\pi+b)\{y^{(j+1)}, x^{(p^s)}y^{(l+1)}\}.$$

We can now prove the main result of this section

Theorem 5.1. *Let $\mathbb{F}$ have odd characteristic p, $q=p^n$, where $n>0$, and let s be a positive integer. Let σ,π be elements in $\mathbb{F}$ such that $\pi^p-\pi=\frac{1}{\sigma^p}$. A graded basis of $H(2;(s+1,n);\Phi(1))$ over the integers modulo $(q-1)p^{s+1}$ is given by the elements*

$$e_{j,k,a} = (1+\sigma x^{(p^s)})^{-j\pi+a}x^{(k+1)}y^{(j+1)},$$

for $j=-1,\cdots q-2$, $k=-1,\cdots p^s-2$ and $a\in\mathbb{F}_p$. The degree of $e_{j,k,a}$ is given by $-(ap^s+k)(q-1)-j \mod p^{s+1}(q-1)$. The corresponding loop algebra L is thin with second diamond in degree q. The diamonds occur in all degrees congruent to 1 modulo $q-1$, that is in all degrees $t(q-1)+1$. If $t\not\equiv 1 \mod p^s$ the corresponding diamond is of infinite type, while for $t\equiv 1 \mod p^s$ the corresponding diamond has finite type. The finite types of the diamonds follow an arithmetic progression not contained in the prime field. The type of the finite diamond in degree $(p^s+1)(q-1)+1$ is $\bar\mu=-1+\frac{1}{\pi}$.

Conversely if the type $\bar\mu$ of the diamond in degree $(p^s+1)(q-1)+1$ is assigned, with $\bar\mu\in\mathbb{F}\setminus\mathbb{F}_p$, then σ and π are uniquely determined by

$$\pi^p-\pi=\frac{1}{\sigma^p}, \quad \bar\mu=-1+\frac{1}{\pi}.$$

Proof. Set $X=e_{-1,0,0}=(1+\sigma x^{(p^s)})^{\pi}x$ and $Y=e_{q-2,-1,0}=(1+\sigma x^{(p^s)})^{2\pi}\bar y$, where X and Y span the homegeneous component of degree 1 of H. Consider the tensor product of H by a polynomial ring $\mathbb{F}[t]$, then L is the subalgebra of $H\otimes\mathbb{F}[t]$ generated by the elements $X\otimes t$ and $Y\otimes t$. It is sufficient (and convenient) to prove the theorem working inside the Hamiltonian algebra H rather than in its loop algebra L, thus simplifying the notation. We already computed the multiplication table of the elements $e_{j,k,a}$ in equations 5.2 and 5.3. It is therefore immediate to describe the adjoint action of X and Y on H. Namely we have

$$\begin{aligned}
&\{e_{j,k,a}, Y\}=0 && \text{for } j\neq -1,0\\
&\{e_{j,k,a}, X\}=e_{j-1,k,a} && \text{for } j\neq -1,0\\
&\{e_{-1,k,a}, X\}=0.
\end{aligned}$$

In particular all the one-dimensional homogeneous components are centralized by Y, except those of degree a multiple of $(q-1)$, and the covering property holds

for these components. The element $v = e_{0,-1,a} = (1+\sigma x^{(p^s)})^a y$, $a \in \mathbb{F}_p$, spans the homogeneous component just before a diamond of finite type. Indeed

$$\begin{aligned}
\{v, X\} &= e_{-1,-1,a} \\
\{v, Y\} &= \sigma(2\pi + a)e_{q-2,p^s-2,a-1} \\
\{v, X, Y\} &= -\sigma(\pi + a)e_{q-3,p^s-2,a-1} \\
\{v, Y, X\} &= \sigma(2\pi + a)e_{q-3,p^s-2,a-1} \\
\{v, X, X\} &= 0 = \{v, Y, Y\}
\end{aligned}$$

We have found that $\mu[v, Y, X] = (1-\mu)[v, X, Y]$ where $\mu = -a\bar{\mu} - a - 1$ as desired. In particular the element $e_{0,-1,0}$ in degree $q-1$ is just before the second diamond of type -1 and the element $e_{0,-1,-1}$ has degree $(p^s+1)(q-1)$, and it is just before the diamond of type $\bar{\mu}$. We will next show that the elements $w = e_{0,k,a} = (1+\sigma x^{(p^s)})^a x^{(k+1)} y$, $k = 0, \dots, p^s - 2$ span the homogeneous components just before the diamonds of infinite type. Indeed

$$\begin{aligned}
[w, X] &= e_{-1,k,a} \\
[w, Y] &= -e_{q-2,k-1,a} \\
[w, X, Y] &= -e_{q-3,k-1,a} \\
[w, Y, X] &= e_{q-3,k-1,a} \\
[w, X, X] &= 0 = [w, Y, Y].
\end{aligned}$$

To complete the proof note that equation 5.2 implies that the element of degree zero in the grading, $e_{0,0,0} = xy$, acts semismply in the ajoint representation on H, since $\{e_{0,0,0}, e_{l,h,b}\} = (h-l)e_{l,h,c}$. □

Remark 5.2. We can interpret Theorem 5.1 in even characteristic. As mentioned in Section 4 the algebra $H(2; (s+1, n); \Phi(1))$ is not simple, but its derived subalgebra $H(2; (s+1, n); \Phi(1))^{(1)}$ is and it has dimension $2^{s+n+1} - 1$. Note first that in odd characteristic the only basis elements which involve the monomial $\bar{x}\bar{y}$ are the elements $e_{q-2,p^s-2,a}$, $a \in \mathbb{F}_p$. However in characteristic 2, the element $e_{q-2,2^s-2,0}$ turns out to be $x^{(p^s-1)}\bar{y}$ and so the only basis element involving $\bar{x}\bar{y}$ is $e_{q-2,2^s-2,1} = x^{(p^s-1)}\bar{y} + \sigma\bar{x}\bar{y}$. The elements $X = e_{-1,0,0}$ and $Y = e_{q-2,-1,0}$ generate $H(2; (s+1, n); \Phi(1))^{(1)}$ and the second diamond in degree $q-1$ becomes fake of type $1 \equiv -1 \bmod 2$. In fact the homogeneous component of degree $q-1$ is generated by $v = e_{0,-1,0}$ and $\{v, Y\} = 2\pi\sigma e_{q-2,2^s-2,1} = 0$.

Remark 5.3. When $s = 0$ the loop algebra of $H(2 : (1, n); \Phi(1))$ according to the grading given in Theorem 5.1 is a Nottingham algebra with all diamonds of finite type. This algebra is constructed in [AM07] as a loop algebra of $H(2 : (1, n); \Phi(1))$ with respect to a certain grading, which differs from the grading given here. We give a brief explanation for this discrepancy. The grading constructed in this section relies on the property of Laguerre polynomials of a derivation D of sending

a grading of H into another grading of H. On the subalgebra $K = \ker(D^p)$ of H the Laguerre polynomial of D we are applying coincides with the exponential map of D, $\exp(D) = \sum_{i=0}^{p-1} \frac{D^i}{i!}$. In the present situation the subalgebra $W = \langle x^{(i+1)}y : i = -1, \dots p^{s+1}-2\rangle$ of $H(2 : (s+1, n); \Phi(1))$ is isomorphic to a Zassenhaus algebra $W(1 : s+1)$ via $E_i = x^{(i+1)}y$ and W is contained in the kernel of D^p. Therefore we are giving to the Zassenhaus algebra $W(1 : m)$ a certain $\mathbb{Z}/p^m\mathbb{Z}$-grading by means of an application of the exponential of D (see [Mat05]). However the Zassenhaus algebra $W(1 : m)$ possesses also a grading over the additive group of the field $\mathbb{F}_{p^m}$. This is just the Cartan decomposition of $W(1 : m)$ with respect to any one-dimensional Cartan subalgebra. There is a standard way of passing from the $\mathbb{Z}$-grading of $W(1 : m)$ to an $\mathbb{F}_{p^m}$-grading, given by the transition formulas

$$\begin{cases} e_0 = E_{-1} + \sigma^{p-1}E_{p^m-2} \\ e_\alpha = \sigma^{p-1}E_{p^m-2} + \sum_{i=-1}^{p^m-2} \alpha^{i+1}E_i \quad \text{for } \alpha \in \mathbb{F}_{p^m}^*\sigma. \end{cases}$$

When $m = 1$ either construction can be used to produce a $\mathbb{Z}/p\mathbb{Z}$ grading of the Witt algebra $W(1 : 1)$. The two gradings thus obtained are conjugated under an automorphism of $W(1 : 1)$, but *they are not the same,* because an element of degree zero in the former is $e_{0,-1,1} = y + \sigma xy = E_{-1} + \sigma E_0$. The latter method was used in [AM07], the former in the present paper, and this accounts for the discrepancy.

We devote the rest of this section to the construction of Nottingham Lie algebras with $p^s - 1$ diamonds of infinite type separated by single occurrences of a diamond of finite type. The types of the latter form an arithmetic progression contained in the prime field. Assume p odd and let $H = H(2; (s+1, n))^{(2)}$ be the Hamiltonian algebra of dimension $p^{s+n+1} - 2$, set $q = p^n$ and $N = p^{s+1}(q-1)$ for some $s > 0$. The derivation $D = (\operatorname{ad} y)^{p^s}$ turns out to be nilpotent on H with $D^p = 0$, since

$$D(x^{(ap^s)}x^{(k+1)}y^{(j+1)}) = x^{((a-1)p^s)}x^{(k+1)}y^{(j+1)}.$$

Thus the exponential of D makes sense on the whole algebra H. Let $\pi \in \mathbb{F}_p$ with $\pi \neq 0$. We obtain a grading of H over $\mathbb{Z}/N\mathbb{Z}$ by assigning to the monomial $x^{(ap^s)}x^{(k+1)}y^{(j+1)}$ degree $-[(a + j\pi)p^s + k](q-1) - j + N\mathbb{Z}$. Since the derivation D is graded of degree N/p, we can apply Theorem 3.1 obtaining a grading of H still over the integers modulo N. Namely we define

$$e_{j,k,a} = (-j\pi + a)! \exp(D)(x^{(-j\pi+a)p^s)}x^{(k+1)}y^{(j+1)}) = (1 + x^{(p^s)})^{-j\pi+a}\, x^{(k+1)}y^{(j+1)}$$

where $0 \le a < p$, $-1 \le k < p^s - 1$ and $-1 \le j < q - 1$, with the exceptions of $(j, k, a) = (-1, -1, a_0)$ and $(j, k, a) = (q-2, p^s-2, a_1)$, where $a_0 \equiv -\pi \bmod p$ and $a_1 \equiv -2\pi - 1 \bmod p$. For later convenience we set $e_{-1,-1,a_0} = 0 = e_{q-2,p^s-2,a_1}$. As in previous section we compute the multiplicative table of the graded elements

$e_{j,k,a}$ obtaining

$$\{e_{j,k,a}, e_{l,h,b}\} = \left(\binom{k+h+1}{h}\binom{j+l+1}{j} - \binom{k+h+1}{k}\binom{j+l+1}{l}\right) e_{j+l,k+h,a+b}, \tag{5.4}$$

for $k+h > -2$, and

$$\{e_{j,-1,a}, e_{l,-1,b}\} = \left(b\binom{j+l+1}{j} - a\binom{j+l+1}{l}\right) e_{j+l,p^s-2,a+b-1}. \tag{5.5}$$

Theorem 5.4. *Let $\mathbb{F}$ have odd characteristic p, and let $q = p^n$, where $n > 0$. Let π be nonzero element in $\mathbb{F}_p$. A graded basis of $H(2;(s+1,n))^{(2)}$ over the integers modulo $(q-1)p^{s+1}$ is given by the elements*

$$e_{j,k,a} = (1 + x^{(p^s)})^{-j\pi+a}\, x^{(k+1)} y^{(j+1)}$$

for $0 \le a < p$, $-1 \le k < p^s - 1$ and $-1 \le j < q-1$. The degree of $e_{j,k,a}$ is given by $-[(a+j\pi)p^s + k](q-1) - j$ The corresponding loop algebra L is thin with second diamond in degree q. The diamonds occur in all degrees congruent to 1 modulo $q-1$, that is in all degrees $t(q-1)+1$. If $t \not\equiv 1 \mod p^s$ the corresponding diamond is of infinite type, while for $t \equiv 1 \mod p^s$ the corresponding diamond has finite type. The finite types of the diamonds follow an arithmetic progression contained in the prime field. The type of the diamond in degree $(p^s+1)(q-1)+1$ is $\bar{\mu} = -1 + \frac{1}{\pi}$.

Conversely, if the type $\bar{\mu}$ of the diamond in degree $(p^s+1)(q-1)+1$ is assigned, with $\bar{\mu} \in \mathbb{F}_p$ and $\bar{\mu} \neq -1$, then π is determined by $\pi(\bar{\mu}+1) = 1$.

Proof. Set

$$X = e_{-1,0,0} = (1 + x^{(p^s)})^{\pi} x \qquad \text{and} \qquad Y = e_{q-2,-1,0} = (1 + x^{(p^s)})^{2\pi} \bar{y}$$

thus X and Y generate the homogeneous component of degree 1 of H. As in previous section we work inside H rather than L. The element $v = e_{0,-1,a}$, $a \in \mathbb{F}_p$, has degree $((p-a)p^s+1)(q-1)$ and generates the homogeneous component preceding a diamond of finite type. Indeed we have

$$\begin{aligned}
\{v, X\} &= e_{-1,-1,a} \\
\{v, Y\} &= (2\pi + a) e_{q-2,p^s-2,a-1} \\
\{v, X, X\} &= 0 = \{v, Y, Y\} \\
\{v, Y, X\} &= (2\pi + a) e_{q-3,p^s-2,a-1} \\
\{v, X, Y\} &= -(\pi + a) e_{q-3,p^s-2,a-1}.
\end{aligned}$$

In particular the element $e_{0,-1,0}$ spans the homogeneous component of degree $q-1$ just preceding a diamond of type -1 and the element $e_{0,-1,-1}$ spans the homogeneous component of degree $(p^s+1)(q-1)$ just preceding a diamond of type $-1+\frac{1}{\pi}$.

The element $w = e_{0,k,a}$, $a \in \mathbb{F}_p$ and $k = 0, \ldots, p^s-2$ has degree $((p-a)p^s-k)(q-1)$ and it is just before a diamond of infinite type. Indeed we have

$$\begin{aligned}
&\{w, X\} = e_{-1,k,a}\\
&\{w, Y\} = e_{q-2,k-1,a}\\
&\{w, X, X\} = 0 = \{w, Y, Y\}\\
&\{w, Y, X\} = e_{q-3,k-1,a}\\
&\{w, X, Y\} = -e_{q-3,k-1,a}.
\end{aligned}$$

The equations $\{e_{j,k,a}, X\} = e_{j-1,k,a}$ for any j and $\{e_{j,k,a}, Y\} = 0$ for $j \neq -1, 0$ complete the proof. $\square$

We observe that the case $\pi = 0$ which is excluded here should correspond to take $\bar{\mu} = \infty$. In this case the grading in Theorem 5.4 does not produce a thin algebra since $\{e_{0,-1,0}, X, Y\} = 0 = \{e_{0,-1,0}, Y, X\}$.

Remark 5.5. Theorem 5.4 can be interpreted for $p = 2$ provided $q > 2$. In this case $\pi = 1$ and the diamonds of finite type are all fake diamonds.

Remark 5.6. Theorem 5.4 can be recovered by Theorem 5.1 in terms of Gerstenhaber's deformation theory of algebras. We refer the reader to [AM07] for a rigorous exposition of the argument. Roughly speaking we can insert a parameter ϵ in equations 4.3 and 4.4 such that as ϵ approaches zero, the type $\bar{\mu}$ in Theorem 5.1 approaches an element in the prime field.

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